# Outperformance and Tracking via Convex Analysis

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## Main Problem

An investor wishes to choose a **portfolio** $\boldsymbol{\pi}$ to invest in and their performance is measured against:

- a **performance benchmark** $\boldsymbol{\rho}$, which the investor wishes to outperform;
- a **tracking portfolio** $\boldsymbol{\eta}$, which the investor penalizes deviations from.

**Main difficulty:**
**Asset growth rates are assumed to be unobservable and/or rank-dependent.**

*What is the optimal portfolio to invest in?*

## Introduction

The portfolio selection problem we consider was discussed in [1], where a **dynamic programming approach** was adopted to arrive at the optimal solution.

### What is new here?

**Asset growth rates** are a typical input for optimal portfolios but are notoriously difficult to estimate robustly. In [1] they are assumed to be **deterministic** functions of time, whereas here we assume they can be:

1. **stochastic**;
2. **unobservable**;
3. **rank-dependent**.

These modeling assumptions should lead to better estimation of asset growth rates and consequently to improved portfolio performance.

### Why convex analysis?

The generalizations above bring **additional difficulties to the dynamic programming approach** (additional state variables, dependence on unobservable processes and local times). However, they can be handled easily by exploiting the convex nature of the stochastic control problem.

## Market Model

**Asset price dynamics:**

$$d\log \underset{n\times 1}{\boldsymbol{X}(t)} = \underset{n\times 1}{\boldsymbol{\gamma}(t)}\,dt + \underset{n\times k}{\boldsymbol{\xi}(t)}\,d\underset{k\times 1}{\boldsymbol{W}(t)}$$

- $\boldsymbol{X}$: *asset prices*
- $\boldsymbol{\gamma}$: *asset growth rates*, driven by an **unobservable Markov chain** $M(t)$ with state space $\mathcal{S}=\{1,2,...,m\}$ and generator matrix $\mathbf{G}$ and dependent on the **rank processes** $X_{(k)}$
- $\boldsymbol{\xi}$: *asset volatilities*, deterministic functions of time
- $\boldsymbol{W}$: standard $k$-dimensional Brownian motion

**Additional terms:**

- $\boldsymbol{\Sigma}$: *asset covariance process*
- $Y^{\boldsymbol{\pi},\boldsymbol{\rho}}$: *log-relative wealth* process of portfolio $\boldsymbol{\pi}$ to portfolio $\boldsymbol{\rho}$.

## Stochastic Control Problem

We wish to find $\boldsymbol{\pi}^*$ which maximizes the **performance criteria**:

$$H(\boldsymbol{\pi}) = \mathbb{E}\Big[\underbrace{\zeta_0\cdot Y^{\boldsymbol{\pi},\boldsymbol{\rho}}(T)}_{①} - \underbrace{\tfrac{\zeta_1}{2}\int_0^T(\boldsymbol{\pi}(s)-\boldsymbol{\eta}(s))'\boldsymbol{\Sigma}(s)(\boldsymbol{\pi}(s)-\boldsymbol{\eta}(s))\,ds}_{②} - \underbrace{\tfrac{\zeta_2}{2}\int_0^T\boldsymbol{\pi}(s)'\mathbf{Q}(s)\boldsymbol{\pi}(s)\,ds}_{③}\Big]$$

The three terms can be interpreted as follows:

1. Terminal reward for **maximizing expected growth rate differential** relative to the performance benchmark $\boldsymbol{\rho}$.
2. Running penalty term **penalizing tracking error/active risk**, i.e. risk-weighted deviations from the tracking benchmark $\boldsymbol{\eta}$.
3. **Quadratic penalty** that can be used to minimize the *absolute* risk of $\boldsymbol{\pi}$ or penalize large positions in certain assets.

## Approach for finding $\boldsymbol{\pi}^*$

1. Compute **posterior probabilities** for Markov chain states
2. **Rewrite** performance criteria in terms of observable processes
3. Compute the **Gâteaux differential** of $H$: $H'$
4. Find portfolio $\boldsymbol{\pi}$ where **Gâteaux differential** vanishes
5. Verify that $H$ is a **strictly concave** functional

## Solution

Let $\mathcal{F}_t$ be the observable filtration and define the posterior probabilities:

$$p_j(t) = \mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{M(t)=j\}}\,\middle|\,\mathcal{F}_t\right] = \frac{P_j(t)}{\sum_{i=1}^m P_i(t)}$$

### Main filtering results

1. The state variables $\{P_j\}_{j=1}^m$ satisfy:
$$\frac{dP_j(t)}{P_j(t^-)} = \sum_{i=1}^m \frac{P_i(t^-)}{P_j(t^-)}\mathbf{G}_{ji}\,dt + \boldsymbol{\lambda}^{(j)}(t^-)'\,d\widetilde{\boldsymbol{W}}(t)$$
where $\underset{k\times 1}{\boldsymbol{\lambda}^{(j)}}(t) = \boldsymbol{\xi}(t)'\boldsymbol{\Sigma}^{-1}(t)\boldsymbol{\gamma}^{(j)}(t)$ and $\boldsymbol{\gamma}^{(j)}$ is the growth rate process when $M$ stays in state $j$.
2. Define $\widehat{\boldsymbol{\gamma}}(t) = \sum_{j=1}^m p_j(t)\boldsymbol{\gamma}^{(j)}(t)$ to be the **posterior mean of asset growth rates**. Then the **innovations process**
$$\widetilde{\boldsymbol{W}}(t) = \int_0^t \boldsymbol{\xi}(u)'\boldsymbol{\Sigma}^{-1}(u)\left(\boldsymbol{\gamma}(u)-\widehat{\boldsymbol{\gamma}}(u)\right)du + \boldsymbol{W}(t)$$
is an $\mathcal{F}_t$-adapted $\mathbb{P}$-Wiener process.

These results allow us to rewrite $H$ in terms of observable processes.

### Optimal control

The *unique* portfolio at which the Gâteaux differential vanishes is:

$$\boldsymbol{\pi}^*_\zeta(t) = \mathbf{A}^{-1}(t)\cdot\left[\frac{1-\mathbf{1}'\mathbf{A}^{-1}(t)\cdot\boldsymbol{B}(t)}{\mathbf{1}'\mathbf{A}^{-1}(t)\mathbf{1}}\cdot\mathbf{1} + \boldsymbol{B}(t)\right]$$

where
$$\mathbf{A}(t) = (\zeta_0+\zeta_1)\boldsymbol{\Sigma}(t) + \zeta_2\mathbf{Q}(t)$$
$$\boldsymbol{B}(t) = \zeta_0\left(\widehat{\boldsymbol{\gamma}}(t) + \tfrac{1}{2}\mathrm{diag}(\boldsymbol{\Sigma}(t))\right) + \zeta_1\boldsymbol{\Sigma}(t)\boldsymbol{\eta}(t)$$

The optimal solution can also be written as the *posterior mean of optimal portfolios in each state*:

$$\boldsymbol{\pi}^*_\zeta(t) = \sum_{i=1}^m p_i(t)\boldsymbol{\pi}^{(i)}_\zeta(t)$$

## References


[1] Ali Al-Aradi and Sebastian Jaimungal. Outperformance and tracking: Dynamic asset allocation for active and passive portfolio management. *arXiv preprint arXiv:1803.05819*, 2018.

[2] Philippe Casgrain and Sebastian Jaimungal. Algorithmic trading with partial information: A mean field game approach. *arXiv preprint arXiv:1803.04094*, 2018.

[3] Ivar Ekeland and Roger Témam. *Convex analysis and variational problems*, volume 28. Siam, 1999.